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NOTES ON COMBINATORIAL MATHEMATICS: ANTI-BLOCKING POLYHEDRA

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PREFACE

Combinatorial mathematics is concerned with arrangements, configurations, relations, and constructions in finite or discrete systems. Combinatorial problems therefore abound in all branches of science and mathematics. Combinatorial approaches—linear and integer programming, network flows, graph theory, and so on—are used much more than they once were, partly because of the availability of high-speed computers. For instance, the practical value of network programming and scheduling algorithms, developed at Rand to deal with Air Force problems over the past decade and now extensively applied to calculating flows through transportation networks, minimum time required to complete projects, and optimal assignments of machines to tasks, is enhanced by the computer's ability to produce numerical answers to very large problems. Another aspect of modern military technology that has focused attention on combinatorics is digital communications, which has necessitated work in error-correcting codes.

Combinatorics is very much problem-oriented but, like all mathematics, it must be carried out at a certain level of abstraction to be worthwhile. For example, there is no permanent value in calculating the capacity of a single, given network, no matter how great the short-run value of the calculation; however, there is permanent value in

devising a good method for calculating the capacity of any network. This process of abstraction occasionally makes the connection between mathematical research and the "real world" somewhat remote, but is absolutely essential if the research is to achieve its maximum utility.

Other Rand publications in combinatorial mathematics will be found in a bibliography of Rand studies on Research in Combinatorics (SB-1030) available on request from the Reports Department, The Rand Corporation, 1700 Main Street, Santa Monica, California 90406.

SUMMARY

A theory parallel to that for blocking pairs of polyhedra is developed for anti-blocking pairs of polyhedra, and certain combinatorial results and problems are discussed in this framework.

Blocking pairs of polyhedra are intimately related to maximum packing problems, anti-blocking pairs to minimum covering problems.

Let $\mathscr{P} = \{x \in R^n_+ | A x \leq 1\}$, where A is a nonnegative matrix and $1 = (1, \ldots, 1)$. The anti-blocker of the convex polyhedron $\mathscr{P}$ is defined to be the convex polyhedron $\bar{\mathscr{P}} = \{x \in R^n_+ | x \cdot \mathscr{P} \leq 1\}$. It is shown that $\bar{\bar{\mathscr{P}}} = \mathscr{P}$ and a method is described for finding a nonnegative matrix B such that $\bar{\mathscr{P}} = \{x \in R^n_+ | B x \leq 1\}$. In particular, if A is the incidence matrix of a family of subsets of $\{1, \ldots, n\}$ having the property that each subset of a member of the family is again a member of the family, a method is described for finding the facets of the convex hull of the rows of A.

It is shown that anti-blocking pairs are characterized by a min-max equality, the analogue of the max-flow min-cut equality for blocking pairs, or by a max-max inequality, the analogue of the length-width inequality for blocking pairs.

Finally, the theory of anti-blocking pairs is applied to certain problems in extremal combinatorics. A main

result is the following. If A and B are an anti-blocking pair of (0,1)-matrices, then the min-max equality holds strongly for both ordered pairs A, B and B, A, i.e., both covering problems $y A \geq w$, $y \geq 0$, min $1 \cdot y$, and $y B \geq w$, $y \geq 0$, min $1 \cdot y$, have integer solutions y for all integer vectors w. This theorem bears on a well-known conjecture in graph theory, called the perfect graph conjecture, and in fact establishes what one might call the pluperfect graph theorem.

CONTENTS

ANTI-BLOCKING POLYHEDRA

1. INTRODUCTION

In [10] the notion of a blocking pair of polyhedra was introduced, and some applications of the resulting theory to extremal combinatorics were described. In this paper we develop a parallel theory for anti-blocking pairs of polyhedra, and discuss certain combinatorial results and problems from this viewpoint.

Blocking pairs of polyhedra have relevance for maximum packing problems, anti-blocking pairs for minimum covering problems. Here by a maximum packing problem we mean the following. Let A be an m by n nonnegative matrix, and let w be a nonnegative n-vector. A solution m-vector y to the linear program

(1.1)
$$\begin{aligned} yA &\leq w \\ y &\geq 0 \\ \max\ &1\cdot y\,, \end{aligned}$$

where $0 = (0, \ldots, 0)$ and $1 = (1, \ldots, 1)$, is a <u>maximum packing</u> in w of the rows of A. Similarly a solution m-vector y to the linear program

(1.2)
$$\begin{aligned} yA &\geq w \\ y &\geq 0 \\ \min\ &1\cdot y\,, \end{aligned}$$

is a <u>minimum covering</u> of w by the rows of A. Normally the words "packing" and "covering" refer to combinatorial situations in which A is a (0,1)-matrix, thought of as the incidence matrix of a family of subsets of $\{1,2, \ldots, n\}$, w is an integer vector (usually $w = 1$), and the solution vector y is required to have integer components, i.e., the maximum in (1.1), or the minimum in (1.2), is taken over all integer vectors y that satisfy the constraints. It is generally an enormous simplification in this situation to drop the integer requirement on y, as we are doing, and to consider merely the real (or rational) packing and covering problems (1.1) and (1.2).

Dual to (1.1) is the linear program

$$\text{(1.3)} \qquad \begin{aligned} A\,x &\geq 1 \\ x &\geq 0 \\ \min\ & w \cdot x\ . \end{aligned}$$

Similarly the dual of (1.2) is

$$\text{(1.4)} \qquad \begin{aligned} A\,x &\leq 1 \\ x &\geq 0 \\ \max\ & w \cdot x\ . \end{aligned}$$

The constraints in (1.3) define an unbounded, n-dimensional, convex polyhedron

$$\text{(1.5)} \qquad \mathcal{C} = \{x \in R_+^n \mid A\,x \geq 1\}$$

situated in the nonnegative orthant R_+^n of R^n. The constraints in (1.4) define an n-dimensional polyhedron

$$(1.6) \qquad \mathscr{B} = \{x \in R_+^n \mid A\,x \leq 1\},$$

also situated in the nonnegative orthant R_+^n. The class of polyhedra of type (1.6) is the primary object of study in this paper. We shall assume throughout that $\mathscr{B}$ is bounded, i.e., that no column of A consists entirely of zeros. This is not an actual restriction, since (1.2) is infeasible unless components of w corresponding to zero columns of A are also zero, in which case such columns of A can be ignored.

In [10] we investigated the blocking relation for polyhedra of type (1.5), and found that it pairs members of this class. The appropriate analogue for polyhedra of type (1.6) is the anti-blocking relation; it also pairs members of this class (Theorem 2.1). Anti-blocking pairs of polyhedra can be characterized by a min-max equality (Theorem 3.1), the analogue of the max-flow min-cut equality for blocking pairs of polyhedra, or by a max-max inequality (Theorem 3.2), the analogue of the length-width inequality for blocking pairs of polyhedra.

An important class of problems in extremal combinatorics is the following. Let $a^1, \ldots, a^m$ be (0,1)-vectors, thought of as the incidence vectors of a family of m subsets of an n-set. (For example, the vectors $a^1, \ldots, a^m$ might represent

the family of all simple paths joining two terminals of a graph G on n edges, the family of all tours in G, the family of all matchings in G, and so on.) How does one characterize the vectors $a^1, \ldots, a^m$ as the extreme solutions of a system of linear inequalities? If $a^1, \ldots, a^m$ are the incidence vectors of a clutter (no member of the family contains another member), it is shown in [10] that the nontrivial facets of the unbounded polyhedron

$$(1.7) \qquad \hat{C} = \text{conv. hull } (\{a^1, \ldots, a^m\}) + R^n_+$$

are given precisely by the extreme solutions of the system $Ax \geq 1$, $x \geq 0$, where A has rows $a^1, \ldots, a^m$. That is, the pair of polyhedra $\check{\mathcal{C}}$ defined by (1.5) and $\hat{\mathcal{C}}$ defined by (1.7) are a blocking pair. Similarly, we find here (Theorem 2.3) that if $a^1, \ldots, a^m$ are the incidence vectors of a family having the property that each subset of a member of the family is again a member of the family, then the facets of

$$(1.8) \qquad \bar{\mathcal{B}} = \text{conv. hull } (\{a^1, \ldots, a^m\})$$

can be determined from the extreme points of its anti-blocking polyhedron $\mathcal{B} = \{x \in R^n_+ | Ax \leq 1\}$. It is no longer true for anti-blocking pairs that each extreme point of one represents a facet of the other, as is the case for blocking pairs (for example, the origin is an extreme point of $\mathcal{B}$).

From the combinatorial point of view, one interesting result of the paper is contained in Sec. 4, where we discuss anti-blocking pairs of (0,1)-matrices, and prove (Theorem 4.1) that if A and B are such an anti-blocking pair, then the min-max equality holds for both ordered pairs A, B and B, A in a strong, integer form. The connection between Theorem 4.1 and certain well-known combinatorial theorems is discussed in Sec. 5, where we note also the connection between Theorem 4.1 and the perfect graph conjecture.

2. THE ANTI-BLOCKING RELATION

Let A be an m by n nonnegative matrix. We assume that $m \geq 1$ and that no column of A consists entirely of zeros. Let

$$\mathcal{B} = \{b \in R^n_+ | Ab \leq 1\} \quad . \tag{2.1}$$

Thus $\mathcal{B}$ is bounded and hence can be written as the convex hull of its extreme points $b^1, \ldots, b^r$:

$$\mathcal{B} = \text{conv. hull } (\{b^1, \ldots, b^r\}). \tag{2.2}$$

It is a consequence of the Farkas lemma on systems of linear inequalities that a row vector a^i of the matrix A is inessential in defining $\mathcal{B}$ if and only if a^i is dominated by a convex combination of other rows of A, i.e., if and only if the inequality

$$a^i \leq \sum_{j=1}^{m} \alpha_j a^j \tag{2.3}$$

holds for some $\alpha_1 \geq 0, \ldots, \alpha_m \geq 0$ satisfying $\alpha_i = 0, \sum_{j=1}^{m} \alpha_j = 1$. Let

$$\bar{\mathcal{B}} = \{a \in R^n_+ | a \cdot \mathcal{B} \leq 1\}. \tag{2.4}$$

We call $\bar{\mathcal{B}}$ the <u>anti-blocker</u> of $\mathcal{B}$.

<u>THEOREM 2.1</u>. <u>Let</u> A <u>be a nonnegative matrix having no zero columns and suppose</u> $\mathcal{B} = \{b \in R^n_+ \mid A\,b \leq 1\}$ <u>has extreme points</u> $b^1, \ldots, b^r$. <u>Let matrix</u> B <u>have rows</u> $b^1, \ldots, b^r$. <u>Then</u> B <u>is nonnegative, has no zero columns, and</u>

$$\bar{\mathcal{B}} = \{a \in R^n_+ \mid B\,a \leq 1\}, \tag{2.5}$$

$$\bar{\bar{\mathcal{B}}} = \mathcal{B}. \tag{2.6}$$

<u>Proof.</u> Clearly B is nonnegative. If the largest element in the i-th column of A is $\mu_i > 0$, then B has as one of its rows the vector $(0, \ldots, 0, 1/\mu_i, 0, \ldots, 0)$, the number $1/\mu_i$ occurring in the i-th position. In particular, B has no zero columns.

Suppose $a \in \bar{\mathcal{B}} = \{a \in R^n_+ \mid a \cdot \mathcal{B} \leq 1\}$. Then $a \cdot b^j \leq 1$, $j = 1, \ldots, r$, and hence $\bar{\mathcal{B}} \subseteq \{a \in R^n_+ \mid B\,a \leq 1\}$. Conversely, suppose $a \in R^n_+$ and $a \cdot b^j \leq 1$ for $j = 1, \ldots, r$. Let $b \in \mathcal{B}$. Thus $b = \sum_{j=1}^{r} \alpha_j b^j$ where $\alpha_j \geq 0$, $\sum_{j=1}^{r} \alpha_j = 1$, and hence

$$a \cdot b = \sum_{j=1}^{r} \alpha_j (a \cdot b^j) \leq 1.$$

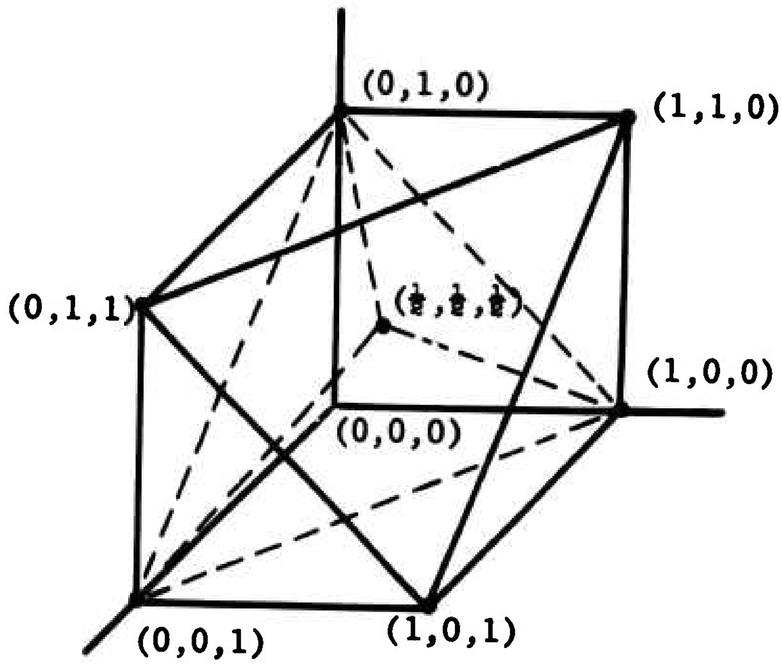
Hence $a \in \bar{\mathcal{B}}$, and (2.5) holds.

Clearly $\mathcal{B} \subseteq \bar{\bar{\mathcal{B}}}$. Suppose $x \in \bar{\bar{\mathcal{B}}}$, $x \notin \mathcal{B}$. Thus for some row of A, say a^1, we have $a^1 \cdot x > 1$, since $x \in R^n_+$, $x \notin \mathcal{B}$. But $a^1 \in R^n_+$ and satisfies $Ba^1 \leq 1$, and so $a^1 \in \bar{\mathcal{B}} = \{a \in R^n_+ | Ba \leq 1\}$. Since $x \in \bar{\bar{\mathcal{B}}}$ and $a^1 \in \bar{\mathcal{B}}$, we must have $a^1 \cdot x \leq 1$, a contradiction. Thus $\bar{\bar{\mathcal{B}}} = \mathcal{B}$.

It follows from Theorem 2.1 that if we are given the matrix A defining $\mathcal{B}$, then a matrix B defining $\bar{\mathcal{B}}$ can be determined as follows. Append the n by n identity matrix to A, and then find an n by n nonsingular submatrix $\bar{A}$ of the matrix thus obtained. Next solve the linear system of equations having $\bar{A}$ as coefficient matrix and having right hand side 1 or 0 according as the corresponding row of $\bar{A}$ belongs to A or to the appended identity. If the resulting solution b satisfies $b \geq 0$, $Ab \leq 1$, then b is an extreme point of $\mathcal{B}$. All extreme points of $\mathcal{B}$ can be obtained in this way.

An example illustrating Theorem 2.1 in R^3 is shown in Fig. 1 below.

In the example, if we start with the matrix A, all of whose rows are essential for $\mathcal{B}$ (define facets of $\mathcal{B}$), we obtain B by the process outlined above. All rows of B except the first are essential for $\bar{\mathcal{B}}$. On the other hand, if we start with B (or just the essential rows of B) and compute the extreme points of $\bar{\mathcal{B}}$, we obtain, in addition to the rows of A, the four vectors (0,0,0), (1,0,0), (0,1,0), (0,0,1) all of which are of course inessential for $\mathcal{B}$. Note in



$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Fig. 1

either case that an inessential extreme point (say an extreme point of $\bar{\mathcal{B}}$ that does not represent a facet of $\mathcal{B}$) is a projection of some other essential extreme point (an extreme point of $\bar{\mathcal{B}}$ that does represent a facet of $\mathcal{B}$). We now prove that this is true in general.

THEOREM 2.2. *Let* A *be a nonnegative matrix defining the polyhedron* $\mathcal{B} = \{b \in R^n_+ | A\,b \leq 1\}$ *and let* $b, b^1, \ldots, b^s$ *be points of* $\mathcal{B}$ *such that* b *is an extreme point of* $\mathcal{B}$ *and is dominated by a convex combination of* $b^1, \ldots, b^s$. *Then* b *is a projection of some* b^i.

Proof. We may suppose

$$b \leq \sum_{i=1}^{k} \alpha_i b^i = c, \tag{2.7}$$

where $\alpha_i > 0$, $i = 1, \ldots, k$, and $\sum_{i=1}^{k} \alpha_i = 1$. If equality holds in (2.7), then, since b is extreme, we have $b = b^1 = \cdots = b^k$, and the conclusion of the theorem holds. Let $b = (\beta_1, \ldots, \beta_n)$, $c = (\gamma_1, \ldots, \gamma_n)$. If $b = 0$, we are done. Rearranging coordinates if necessary, we may now suppose that

$$\beta_1 > 0, \ldots, \beta_e > 0, \beta_{e+1} = \cdots = \beta_n = 0.$$

Since b is extreme in $\mathcal{B}$, we can find an e by e submatrix E of A such that the equations $E\,x = 1$ have the unique solution

$x = (\beta_1, \ldots, \beta_e)$. Let $y = (\gamma_1, \ldots, \gamma_e)$. Since $0 \leq x \leq y$ and $E \geq 0$, we have $E\,y \geq 1$. If some component of Ey is greater than 1, then $c \notin \mathscr{B}$, a contradiction. Hence $E\,y = 1$, and thus, since E is nonsingular, $y = x$. Let $b^i = (\beta_1^i, \ldots, \beta_n^i)$, and define projections $b_*^i = (\beta_1^i, \ldots, \beta_e^i, 0, \ldots, 0)$. Since $\beta_i = \gamma_i$ for $i = 1, \ldots, e$ and $\beta_{e+1} = \cdots = \beta_n = 0$, we have

$$b = \sum_{i=1}^{k} \alpha_i b_*^i ,$$

where $b_*^i \in \mathscr{B}$, $i = 1, \ldots, k$. Because b is extreme in $\mathscr{B}$, it follows that $b = b_*^1 = \cdots = b_*^k$, and hence b is a projection of b^i for $i = 1, \ldots, k$.

Theorem 2.2 is useful in various ways. For example, if we are given the nonnegative matrix A defining $\mathscr{B} = \{b \in R_+^n | A\,b \leq 1\}$ and are able to find the extreme points of $\mathscr{B}$, then the facets of the anti-blocking polyhedron $\bar{\mathscr{B}}$ can be determined easily, since each facet of $\bar{\mathscr{B}}$ corresponds to an extreme point of $\mathscr{B}$ that is not a projection of some other extreme point of $\mathscr{B}$. Another use is in the proof of Theorem 2.3 below.

<u>THEOREM 2.3.</u> <u>Let</u> $a^1, \ldots, a^m$ <u>be the incidence vectors of a clutter of</u> m <u>subsets</u> $S_1, \ldots, S_m$ <u>of</u> $\{1, \ldots, n\}$, <u>and let</u> A <u>have rows</u> $a^1, \ldots, a^m$. <u>Let</u> $\mathscr{B} = \{b \in R_+^n | A\,b \leq 1\}$

be bounded with extreme points $b^1, \ldots, b^r$ *and let the matrix* B *have rows* $b^1, \ldots, b^r$. *Then the extreme points of the bounded polyhedron* $\bar{\mathcal{B}} = \{a \in R^n_+ | B\,a \leq 1\}$ *are precisely* $a^1, \ldots, a^m$ *together with all incidence vectors* of subsets of $S_1, \ldots, S_m$ (*i.e., all projections of* $a^1, \ldots, a^m$).

Proof. Since A is the incidence matrix of a clutter, no row of A is dominated by a convex combination of other rows of A, and hence each row of A is essential for $\mathcal{B}$. Consequently, by Theorem 2.1, each row of A is an extreme point of $\bar{\mathcal{B}}$. Moreover, since B contains the n by n identity matrix as a submatrix, it follows that the incidence vector of a subset of any S_i is also an extreme point of $\bar{\mathcal{B}}$. There can be no others, for if a is an extreme point of $\bar{\mathcal{B}}$ that is inessential for $\mathcal{B}$, then a is dominated by a convex combination of rows of A, and hence by Theorem 2.2, a is a projection of some a^i.

The example of Fig. 1 illustrates Theorem 2.3. The extreme points of $\bar{\mathcal{B}} = \{a \in R^n_+ | B\,a \leq 1\}$ are precisely the rows of A (the incidence vectors of the clutter of all 2-sets of a 3-set) together with the incidence vectors of all singletons and the empty set.

In the rest of this section we discuss a connection between anti-blocking pairs of polyhedra and blocking pairs of polyhedra [10]. We describe this connection in the

context of (0,1)-matrices, and shall show (Theorem 2.4) that if A is the incidence matrix of a clutter, if B is the blocking matrix of A, and if A' is the complement of A (i.e., A' is obtained from A by interchanging 0's and 1's) then the anti-blocking polyhedron of the polyhedron $\mathcal{B}' = \{b \in R_+^n | A'b \leq 1\}$ can be obtained easily from the matrix B.

We recall from [10] that the blocking polyhedron of the (unbounded) polyhedron $\mathcal{B} = \{b \in R_+^n | A\,b \geq 1\}$, is the (unbounded) polyhedron $\hat{\mathcal{B}} = \{a \in R_+^n | a \cdot \mathcal{B} \geq 1\}$, and that the nontrivial facets of $\hat{\mathcal{B}}$ correspond precisely to the extreme points of $\mathcal{B}$, i.e., if $\mathcal{B}$ has extreme points $b^1, \ldots, b^r$ and if B is the matrix having rows $b^1, \ldots, b^r$, then $\hat{\mathcal{B}} = \{a \in R_+^n | B\,a \geq 1\}$, and each row of B is essential in defining $\hat{\mathcal{B}}$. If A is a (0,1)-matrix and if each row of A is essential in defining $\mathcal{B}$, then A is the incidence matrix of a clutter; in this case the blocking matrix B contains as a submatrix the incidence matrix of the blocking clutter [8,13], i.e., B has a row corresponding to each (0,1)-vector that has inner product at least 1 with all rows of A, and is minimal with respect to this property. In general, B will have many other fractional rows in addition to these integer rows.

THEOREM 2.4. *Let* A *be the* m *by* n *incidence matrix of a clutter on* $\{1, \ldots, n\}$, *and suppose* A *has no column consisting entirely of 1's. Let* B *be the* r *by* n *blocking matrix of* A, *and let* ρ_j *denote the sum of the elements in the* j-th *row* b^j *of* B. *Let* A' *be the complement of* A. *Then the anti-blocking polyhedron of* $\mathcal{B}' = \{b \in R^n_+ | A'b \leq 1\}$ *is the polyhedron* $\bar{\mathcal{B}}' = \{a \in R^n_+ | a \leq 1, B'a \leq 1\}$, *where* B' *is the* r *by* n *matrix having rows* $b^1/(\rho_1-1), \ldots, b^r/(\rho_r-1)$.

Proof. We note first that $\rho_j > 1$. For, since A is a (0,1)-matrix, we surely have $\rho_j \geq 1$. If $\rho_j = 1$, then, since b^j is an extreme point of the polyhedron $\mathcal{B} = \{x \in R^n_+ | Ax \geq 1\}$, it follows that b^j is the incidence vector of a singleton, and hence the j-th column of A consists entirely of 1's, contradicting our assumption on A.

We next prove a lemma.

LEMMA. *Let* E *be an* e *by* e *nonsingular* (0,1)-*matrix. Suppose the equations* $Ex = 1$ *have the unique solution* $x = (\xi_1, \ldots, \xi_e)$, *and that* $x \geq 0$, $\sum_{i=1}^{e} \xi_i > 1$. *Let* E' *be the complement of* E. *Then the equations* $E'y = 1$ *have a unique solution* $y = (\eta_1, \ldots, \eta_e)$. *Moreover*, $y \geq 0$ *and* $\sum_{i=1}^{e} \eta_i > 1$.

Let $\rho = \sum_{i=1}^{e} \xi_i$, and let J denote the e by e matrix consisting entirely of 1's. Then $y = x/(\rho-1)$ satisfies

$$E'y = (J-E)\left(\frac{x}{\rho-1}\right) = \frac{Jx}{\rho-1} - \frac{Ex}{\rho-1}$$

$$= \left(\frac{\rho}{\rho-1}, \ldots, \frac{\rho}{\rho-1}\right) - \left(\frac{1}{\rho-1}, \ldots, \frac{1}{\rho-1}\right)$$

$$= 1.$$

Clearly $y \geq 0$ and has component sum $\sigma = \frac{\rho}{\rho-1} > 1$. If $E'y = 1$ has two distinct solutions y_1 and y_2, with component sums $\sigma_1 \neq 1$, $\sigma_2 \neq 1$, we deduce as above that $Ex = 1$ has two distinct solutions, contradicting our assumption. If $E'y = 1$ has a solution y with component sum $\sigma = 1$, then $Ey = (J-E')y = Jy - E'y = 1-1 = 0$, and hence E is singular, again a contradiction. This proves the lemma.

Since b^j is an extreme point of $\mathcal{B} = \{x \in R^n_+ | Ax \geq 1\}$, there is a nonsingular submatrix E of A such that the nonzero coordinates of b^j are given as the solution of the equations $Ex = 1$. Applying the lemma, we see that $b^j/(\rho_j-1)$ is an extreme point of $\mathcal{B}' = \{a \in R^n_+ | A'b \leq 1\}$ if this vector satisfies all the inequalities defining $\mathcal{B}'$. This follows as in the proof of the lemma, since

$$A' \frac{b^j}{\rho_j-1} = (J-A) \frac{b^j}{\rho_j-1} = \frac{Jb^j}{\rho_j-1} - \frac{Ab^j}{\rho_j-1}$$

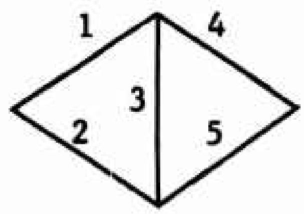
$$\leq \left(\frac{\rho_j}{\rho_j-1}, \ldots, \frac{\rho_j}{\rho_j-1}\right) - \left(\frac{1}{\rho_j-1}, \ldots, \frac{1}{\rho_j-1}\right)$$

$$\leq 1.$$

In an exactly similar way, we see that an extreme point b of $\mathcal{B}'$ has component sum $\sigma \geq 1$, and that if $\sigma > 1$, the same transformation $b \rightarrow b/(\sigma-1)$ produces an extreme point of $\mathcal{B}$. Since A' has no columns of zeros, each unit vector is also an extreme point of $\mathcal{B}'$, and these are the only extreme points of $\mathcal{B}'$ having component sums equal to 1. This completes the proof of Theorem 2.4.

It follows from Theorems 2.3 and 2.4 that if we know inequalities that characterize the incidence vectors of a clutter as the extreme points of a polyhedron of type (1.5), then we know inequalities that characterize the convex hull of all incidence vectors of the family consisting of the complementary clutter plus subsets of members of this clutter.

We conclude this section with an example illustrating Theorem 2.4. In Fig. 2 below, the matrix A is the incidence matrix of all spanning trees of the graph shown there, B is the blocking matrix of A, A' is the incidence matrix of all cotrees, and B' is obtained from B as in Theorem 2.4. Inessential rows of B' have a line drawn through them. As the example indicates, much simplification can occur in passing from B to B'.



$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} (\geq 1) \qquad B' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \cancel{\frac{1}{2}} & \cancel{0} & \cancel{\frac{1}{2}} & \cancel{0} & \cancel{\frac{1}{2}} \\ \cancel{0} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{0} \\ \cancel{0} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{0} & \cancel{\frac{1}{2}} \\ \cancel{\frac{1}{2}} & \cancel{0} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{0} \\ \cancel{0} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} \\ \cancel{\frac{1}{2}} & \cancel{0} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} \\ \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{0} & \cancel{\frac{1}{2}} \\ \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} & \cancel{0} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} (\leq 1)$$

Fig. 2

3. THE MIN-MAX EQUALITY AND MAX-MAX INEQUALITY

In this section we develop analogues for anti-blocking pairs of the max-flow min-cut equality and the length-width inequality for blocking pairs of polyhedra.

Let A and B be nonnegative matrices, each having n columns and neither having zero columns. Let the rows of A be $a^1, \ldots, a^m$ and the rows of B be $b^1, \ldots, b^r$. We say that the <u>min-max equality</u> holds for the pair A, B (in this order) if and only if, for each $w \in R^n_+$, it is true that in the linear program

$$\begin{aligned} &yA \geq w, \\ &y \geq 0, \\ &\min 1 \cdot y , \end{aligned} \tag{3.1}$$

we have

$$\min 1 \cdot y = \max_{1 \leq j \leq r} b^j \cdot w . \tag{3.2}$$

Similarly, we say that the <u>max-max inequality</u> holds for the pair A, B if and only if, for every $\ell \in R^n_+$, $w \in R^n_+$, we have

$$(\max_{1 \leq i \leq m} a^i \cdot \ell)(\max_{1 \leq j \leq r} b^j \cdot w) \geq \ell \cdot w. \tag{3.3}$$

<u>THEOREM 3.1.</u> <u>The min-max equality holds for the pair</u> A, B <u>if and only if the polyhedra</u>

$\mathcal{B} = \{b \in R^n_+ | A\,b \leq 1\}$ <u>and</u> $\mathcal{A} = \{a \in R^n_+ | B\,a \leq 1\}$ <u>are an anti-blocking pair.</u> <u>Hence if the min-max equality holds for</u> A, B, <u>it also holds for</u> B, A.

<u>Proof.</u> Suppose that $\mathcal{B}$ and $\mathcal{A}$ are an anti-blocking pair. By Theorem 2.1, the matrix B contains as a row each extreme vector of $\mathcal{B}$ that is essential for $\mathcal{A} = \bar{\mathcal{B}}$. Since an inessential row of B can be ignored in computing $\max_{1 \leq j \leq r} b^j \cdot w$, it follows from the linear programming duality theorem, together with the fact that the maximum value of a linear form defined over $\mathcal{B}$ is achieved at an extreme point of $\mathcal{B}$, that the minimum value of $1 \cdot y$ in the linear program (3.1) is equal to

$$\max_{x \in \mathcal{B}} w \cdot x = \max_{1 \leq j \leq r} b^j \cdot w.$$

Conversely, suppose the min-max equality holds for the pair A, B (in this order). Let $\mathcal{B}$ have extreme points $\bar{b}^1, \ldots, \bar{b}^s$, and let the matrix $\bar{B}$ have these as its rows. We shall show that

$$(3.4) \qquad \mathcal{A} = \{x \in R^n_+ | \bar{B}\,x \leq 1\}.$$

Suppose there is a $w \in R^n_+$ such that $B\,w \leq 1$, but $\bar{B}\,w$ has some component greater than 1. Then

$$(3.5) \qquad \max_{1\le j\le r} b^j \cdot w < \max_{1\le i\le s} \bar{b}^i \cdot w = \max_{x\in \mathcal{B}} w \cdot x .$$

But by the min-max equality and the duality theorem for linear programs, we have that min $1\cdot y$, subject to the constraints $yA \ge w$, $y \ge 0$, is equal to both left and right hand sides of (3.5), a contradiction. Hence $\mathcal{A} \subseteq \{x \in R^n_+ | \bar{B}x \le 1\}$. Similarly we see that $\mathcal{A} \supseteq \{x \in R^n_+ | \bar{B}x \le 1\}$. Hence (3.4) holds, and Theorem 2.1 implies that $\mathcal{A} = \bar{\mathcal{B}}$.

<u>THEOREM 3.2.</u> <u>The polyhedra</u> $\mathcal{B} = \{b \in R^n_+ | A b \le 1\}$ <u>and</u> $\mathcal{A} = \{a \in R^n_+ | B a \le 1\}$ <u>are an anti-blocking pair if and only if</u> (i) $a^i \cdot b^j \le 1$ <u>for all</u> $i = 1, \ldots, m$, $j = 1, \ldots, r$, <u>and</u> (ii) <u>the max-max inequality holds for the pair</u> A, B.

<u>Proof.</u> Assume (i) and (ii). (Note that (ii) implies our blanket assumption that no column of A or B is zero.) We show first that

$$(3.6) \qquad \mathcal{B} \subseteq \bar{\mathcal{A}} = \{x \in R^n_+ | x \cdot \mathcal{A} \le 1\},$$

$$(3.7) \qquad \mathcal{A} \subseteq \bar{\mathcal{B}} = \{x \in R^n_+ | x \cdot \mathcal{B} \le 1\}.$$

Suppose $a \in \mathcal{A}$, $b \in \mathcal{B}$. Then

$$\max_{1\le j\le r} b^j . a \le 1, \quad \max_{1\le i\le m} a^i . b \le 1.$$

Hence by (ii),

$$1 \geq (\max_{1\leq i\leq m} a^i \cdot b)(\max_{1\leq j\leq r} b^j \cdot a) \geq a \cdot b.$$

Thus $a \cdot \mathcal{B} \leq 1$ and $b \cdot \mathcal{A} \leq 1$. Hence $a \in \overline{\mathcal{B}}$, $b \in \overline{\mathcal{A}}$, verifying (3.6) and (3.7). If the inclusion in (3.6) is proper, let $\bar{a} \in \overline{\mathcal{A}}$, $\bar{a} \notin \mathcal{B}$. Since $\bar{a} \in \overline{\mathcal{A}}$, we have $\bar{a} \cdot \mathcal{A} \leq 1$. Since $\bar{a} \notin \mathcal{B}$, we have $a^i \cdot \bar{a} > 1$ for some $i = 1, \ldots, m$. But by (i), $a^i \in \mathcal{A}$, a contradiction. Hence $\mathcal{B} = \overline{\mathcal{A}}$.

Conversely, suppose $\overline{\mathcal{A}} = \mathcal{B}$. If $a^i \cdot b^j > 1$ for some i, j, then $a^i \notin \mathcal{A} = \overline{\mathcal{B}}$, and hence $a^i \cdot b > 1$ for some $b \in \mathcal{B}$, contradicting the definition of $\mathcal{B}$. Hence (i) holds. Let $\ell \in R^n_+$, $w \in R^n_+$, and define

$$\lambda = \max_{1\leq i\leq m} a^i \cdot \ell, \tag{3.8}$$

$$\omega = \max_{1\leq j\leq r} b^j \cdot w. \tag{3.9}$$

If either $\ell = 0$ or $w = 0$, the max-max inequality holds trivially. Assume $\ell \neq 0$, $w \neq 0$. Suppose $\lambda = 0$. Then some column of A is zero, contradicting our assumption on A. Hence $\ell \neq 0$ implies $\lambda > 0$. Similarly, $w \neq 0$ implies $\omega > 0$. Then, by (3.8) and (3.9), we have

$$a^i \cdot (\ell/\lambda) \leq 1, \; i = 1, \ldots, m, \tag{3.10}$$

$$b^j \cdot (w/\omega) \leq 1,\ j = 1, \ldots, r, \tag{3.11}$$

and hence $\ell/\lambda \in \mathcal{B} = \bar{\mathcal{a}}$, $w/\omega \in \mathcal{a}$. Consequently $(\ell/\lambda)\cdot(w/\omega) \leq 1$, $\ell \cdot w \leq \lambda\ \omega$.

4. ANTI-BLOCKING PAIRS OF (0,1)-MATRICES.

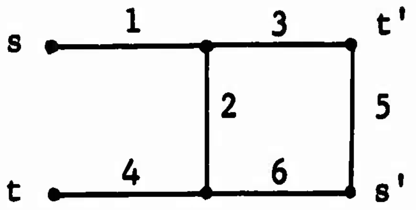
In this section we focus attention on anti-blocking pairs of (0,1)-matrices. There are wide classes of such matrices having special combinatorial interest; some of these will be discussed in the next section.

If A and B are nonnegative matrices that define an anti-blocking pair of polyhedra, we call A, B an *anti-blocking pair* of matrices. (We could of course restrict A and B to essential rows in discussing anti-blocking matrices.) If A is a (0,1)-matrix with anti-blocker B, we say that the min-max equality holds *strongly* for A, B provided the linear program (3.1) has an integer solution vector y whenever w is a nonnegative integer vector. It is intuitively clear that a necessary condition for the strong min-max equality is that all essential rows of B be (0,1)-vectors. It is surprising that this condition is also sufficient.

THEOREM 4.1. *Let* A *be a* (0,1)-*matrix having no zero columns and let* B *be an anti-blocking matrix of* A. *The min-max-equality holds strongly for* A, B *if and only if each essential row of* B *is a* (0,1)-*vector. Hence if the min-max equality holds strongly for* A, B, *it holds strongly for* B, A.

Before proving Theorem 4.1, we emphasize that the analogous statement for blocking pairs of matrices is false. A counterexample for blocking pairs is shown below in Fig. 3. The example is based on the result, due to T. C. Hu [12], that the max-flow min-cut theorem is valid for two-commodity flows in undirected graphs, but that fractional flows may be required. In the example, the matrix A is the incidence matrix of all s to s' and all t to t' paths in the graph shown. Take $w = 1$ and observe that the unique solution to the program $yA \leq w$, $y \geq 0$, max $1 \cdot y$, is given by $y = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. It can also be shown for this example that the program $yB \leq w$, $y \geq 0$, max $1 \cdot y$, always has integer solutions for arbitrary nonnegative integer vectors w. Thus integer solutions in one of the two maximum packing programs for a blocking pair of (0,1)-matrices does not imply integer solutions in the other.

<u>Proof of Theorem 4.1.</u> Suppose the min-max equality holds strongly for A, B, and assume that B has a fractional row $b = (\beta_1, \ldots, \beta_n)$ that is essential. Thus b is an extreme point of the polyhedron $\mathcal{B} = \{x \in R^n_+ | A x \leq 1\}$ and there is a nonnegative integer vector $w = (\omega_1, \ldots, \omega_n)$ such that the maximum value of $w \cdot x$, for $x \in \mathcal{B}$, is achieved uniquely at $x = b$. Let λ be a positive integer. By assumption, the linear program $yA \geq w$, $y \geq 0$, min $1 \cdot y$, has an integer solution y and $(\lambda w) \cdot b = 1 \cdot (\lambda y)$ is a positive



A: $\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$
$$\boxed{\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}}$$

(blocker) B:
$$\boxed{\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}}$$

Fig. 3

integer α. Suppose $0 < \beta_1 < 1$, and let $w_\lambda = (\lambda\omega_1 + 1, \lambda\omega_2, \ldots, \lambda\omega_n)$. Thus $w_\lambda \cdot b = \alpha + \beta_1$ is not an integer. Consider the program $yA \geq w_\lambda$, $y \geq 0$, min $1 \cdot y$. There is an integer vector y_λ solving this program, and this vector y_λ satisfies $1 \cdot y_\lambda = \max w_\lambda \cdot b'$, where the maximum is over rows b' of B. Since $1 \cdot y_\lambda$ is an integer and $w_\lambda \cdot b$ is not an integer, there is a row $b_\lambda = (\beta_1(\lambda), \ldots, \beta_n(\lambda)) \neq b$ of B such that

$$w_\lambda \cdot b_\lambda = \beta_1(\lambda) + (\lambda w) \cdot b_\lambda > w_\lambda \cdot b = \beta_1 + (\lambda w) \cdot b,$$

and hence

$$\beta_1(\lambda) > \lambda(w \cdot b - w \cdot b_\lambda). \tag{4.1}$$

But $w \cdot b > w \cdot b'$ for all rows $b' \neq b$ of B. Hence, since B has finitely many rows, we have $w \cdot b - w \cdot b_\lambda > \delta > 0$, where δ is independent of λ. Thus from (4.1), $\beta_1(\lambda) > \lambda\delta$. Since λ is an arbitrary positive integer and B has finitely many rows, this is absurd. Hence all essential rows of B are (0,1)-vectors.

Suppose, conversely, that each essential row of B is a (0,1)-vector. We shall describe an algorithm for obtaining an integer solution to the linear program $yA \geq w$, $y \geq 0$, min $1 \cdot y$, where w is a nonnegative integer vector.* (Actually, in the description, we suppose the initial w is positive. As will be clear, this is merely a convenience.) We know

*See the proof of Theorem 5 of [1] for a special case of this construction.

that min $1 \cdot y$ subject to these constraints is equal to $\max_j b^j \cdot w$, where $b^1, \ldots, b^r$ are the essential rows of B. Let $\max_j b^j \cdot w = \omega$, a positive integer. Suppose that $b^j \cdot w = \omega$ for $j = 1, \ldots, k$, $b^j \cdot w < \omega$ for $j = k+1, \ldots, r$. Then the linear system of equations and inequalities

$$b^j \cdot x = 1, \quad j = 1, \ldots, k, \tag{4.2}$$

$$b^j \cdot x \leq 1, \quad j = k+1, \ldots, r, \tag{4.3}$$

$$x \geq 0, \tag{4.4}$$

has the solution w/ω. It follows that there is an extreme point a of $\mathcal{A} = \{x \in R_n^+ | Bx \leq 1\}$ that satisfies (4.2)–(4.4), and hence either vector a is a row of A or is a projection of some row of A, say either $a = a^i$ or a is a projection of a^i. Set the i-th component of y equal to 1 (temporarily), reduce all components of w that correspond to positive entries in a^i by 1, and delete columns in A and B that correspond to nonpositive components of the reduced weight vector, as well as these components of the reduced weight vector, obtaining matrices A', B', and an integer vector $w' > 0$. One can verify (either directly or by using Theorem 3.2) that the matrices A' and B' constitute an anti-blocking pair. Moreover, since $b^j \cdot a^i = 1$ for $j = 1, \ldots, k$, and $b^j \cdot a^i = 0$ or 1 for $j = k+1, \ldots, m$, it follows that $\max b' \cdot w'$, taken over all rows b' of B', is equal to $\omega - 1$. We can now repeat

the argument, and in this way build up an integer solution vector y to the program $yA \geq w$, $y \geq 0$, min $1 \cdot y$. This completes the proof of Theorem 4.1.

A key point in the proof (one that breaks down in attempting an analogous argument for blocking pairs of (0,1)-matrices) is that the matrices A' and B' again constitute an anti-blocking pair. Geometrically, deleting column i from A corresponds to projecting the polyhedron $\mathcal{B} = \{x \in R^n_+ | Ax \leq 1\}$ on the hyperplane $\xi_i = 0$, which is the same as intersecting $\mathcal{B}$ with this hyperplane. The dual operation for B is again to delete column i from B, i.e., to project the anti-blocking polyhedron $\bar{\mathcal{B}}$ on the hyperplane $\xi_i = 0$. (For blocking pairs of polyhedra, the dual of a projection of one on the hyperplane $\xi_i = 0$ is not a projection of the other, but is instead an intersection of the latter with this hyperplane. Projections and intersections are distinct operations for blocking pairs. See [10], where these dual operations are called deletions and contractions, in analogy with operations in matroid theory.)

It follows from Theorem 2.1 and well-known linear programming results [11] that if A is a totally unimodular (0,1)-matrix (all square submatrices of A have determinant 0, 1, or -1), then there is an anti-blocking matrix B for A that is also a (0,1)-matrix. Hence the min-max equality holds strongly for both A, B and B, A if one of the two is totally unimodular. (It is far from true that total unimodularity for A implies total unimodularity for B.)

There are significant classes of examples of blocking pairs A, B of (0,1)–matrices where it is trivial to see directly that the min–max equality holds strongly for B, A, say, but where the strong min–max equality for A, B is a substantial theorem.

For example, consider the Dilworth theorem [2] on minimal chain decompositions of partially ordered sets. (This will be discussed in more detail in the next section.) The Dilworth theorem is surely a substantial theorem. The dual theorem (in the anti–blocking sense) is just the statement that a minimal decomposition of a partially ordered set into anti–chains has cardinality equal to the length of a longest chain. But this latter theorem is a triviality, since a minimal decomposition into anti–chains can be obtained by deleting all minimal elements in the partially ordered set, then repeating the process in the reduced partially ordered set, and so on. And yet, in view of Theorem 4.1, these two dual theorems are in a certain sense equivalent.

5. COMBINATORIAL EXAMPLES

In this concluding section we discuss some examples of anti-blocking pairs of polyhedra that have combinatorial interest. In each example we take A to be an m by n (0,1)-matrix, so that A can be viewed as the incidence matrix of a family of m subsets of an n-set, and describe an r by n anti-blocking matrix B for A.

Example 1. (Permutations). Let A be the $m = s!$ by $n = s^2$ (0,1)-matrix having a column for each cell ij of an s by s array and having a row corresponding to each s by s permutation matrix, viewed as a vector in R^n. It is well known that the inequalities

$$(5.1) \qquad \sum_{j=1}^{s} \xi_{ij} \leq 1, \quad i = 1, \ldots, s,$$

$$(5.2) \qquad \sum_{i=1}^{s} \xi_{ij} \leq 1, \quad j = 1, \ldots, s,$$

$$(5.3) \qquad \xi_{ij} \geq 0, \quad \text{all } i, j,$$

have as extreme solutions $x = (\xi_{ij})$ precisely the rows of A together with all projections of these rows. Hence (5.1)-(5.3) define the anti-blocking polyhedron of $\mathcal{B} = \{x \in R^n | Ax \leq 1\}$. In other words, an anti-blocking matrix of A is the $r = 2s$ by $n = s^2$ matrix B whose rows are the incidence vectors of the rows and columns of an s by s array. (See Fig. 4 below for an illustration for $s = 3$.)

11	12	13
21	22	23
31	32	33

$$A: \quad \begin{array}{c} \begin{matrix} 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

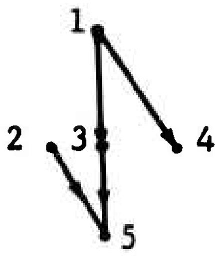
$$B: \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Fig. 4

It is well known that the min-max equality holds strongly for both A, B and B, A. (This also follows from Theorem 4.1 and the fact that B is totally unimodular.) For w a (0,1)-vector, the strong min-max equality for B, A is the classical König theorem on maximum matchings and minimum covers in bipartite graphs, and the strong min-max equality for A, B is the theorem, also due to König, that the minimum number of colors required for an edge-coloring in a bipartite graph is equal to the maximum valence in the graph. Neither theorem is obvious.

Observe that the max-max inequality says here that if ℓ and w are nonnegative s by s matrices, that if λ is the value of an optimum assignment for ℓ (i.e., λ is the largest sum obtainable from ℓ by selecting just one entry from each row and column), and if ω is the largest row or column sum of w, then $\lambda\omega \geq \ell \cdot w$.

<u>Example 2. (Chains in a partially ordered set)</u>. Let A be the incidence matrix of all chains in a partially ordered set on n elements. Here one can deduce, either from the Dilworth theorem [2] on chain decompositions of partially ordered sets or from known results about network flows [9], that an anti-blocking matrix B for A is the matrix whose rows are the incidence vectors of all anti-chains (a subset of elements, no two of which are comparable) of the partially ordered set. (See Fig. 5 below for an illustration, where we have listed only essential rows of A (maximal chains) and of B (maximal anti-chains).)



$$A: \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \boxed{\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}} \end{array}$$

$$B: \boxed{\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}}$$

Fig. 5

For $w = 1$, the strong min-max equality for A, B is the Dilworth theorem; the extension to nonnegative integer vectors w can be deduced from the Dilworth theorem by replicating elements appropriately in the partial ordering. The strong min-max equality for B, A is, on the other hand, a triviality, since the following simple algorithm solves the linear program $yB \geq w$, $y \geq 0$, min $1 \cdot y$. Select the anti-chain of all minimal elements in the partially ordered set, and set the corresponding component of y equal to η, where η is the least of the weights (components of w) assigned to members of this anti-chain. Reduce each of these weights by η, delete any elements now having weight zero, and repeat the procedure.

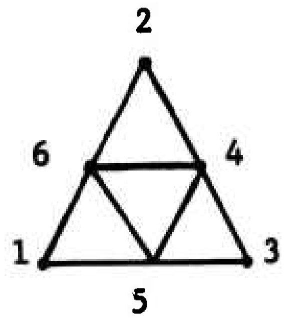
In general, neither the chain matrix A nor the anti-chain matrix B is totally unimodular.

The max-max inequality asserts that if we are given two weight vectors w and ℓ, then the product of the largest chain-weight, computed using ℓ, and the largest anti-chain weight, computed using w, is at least equal to $\ell \cdot w$.

<u>Example 3. (Cliques in graphs).</u> Let A be the incidence matrix of all cliques (a subset of vertices, every pair joined by an edge) in a graph G on n vertices. In general, no decent characterization of an anti-blocking matrix B is known for this situation. But if G is either a rigid circuit graph (every circuit of four or more edges has a chord), a comparability graph (orientations can be assigned the edges

of G so that the resulting directed graph represents a partial order on the vertices of G, i.e., if $a \to b$, $b \to c$, then $a \to c$), or the complement of a rigid-circuit graph or of a comparability graph, then an anti-blocking matrix B has rows that are the incidence vectors of all independent sets of vertices of G. (A set of vertices is independent if no pair is joined by an edge.) (For illustrations see Fig. 6 below, where we have listed maximal cliques and maximal independent sets only, i.e., the essential rows of A and B.)

Note that in complementing the graph G, we interchange the roles of A and B. Hence it suffices to consider only the cases (a) G is a rigid-circuit graph, and (b) G is a comparability graph. The second of these has been dealt with above, since a clique in G is a chain in the resulting partially ordered set, and an independent set is an anti-chain. We shall dispose of (a) by sketching an algorithm which can be used to prove that the min-max equality holds for A, B in the strong form. (It is known [1] that if w is a (0,1)-vector, the integer form of the min-max equality holds, but it does not seem to follow directly from this fact that it also holds for general w. The device of replicating a vertex can destroy the rigid-circuit property.) This algorithm for computing min $1 \cdot y$ subject to $yA \geq w$, $y \geq 0$, is based on the known fact [3] that a rigid-circuit graph always contains a simplical vertex. Here a vertex

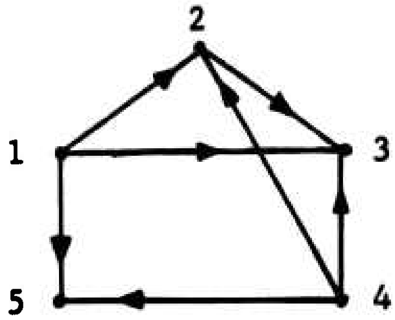


A:

1	2	3	4	5	6
1	0	0	0	1	1
0	1	0	1	0	1
0	0	1	1	1	0
0	0	0	1	1	1

B:

1	1	1	0	0	0
1	0	0	1	0	0
0	1	0	0	1	0
0	0	1	0	0	1



A:

1	2	3	4	5
1	1	1	0	0
0	1	1	1	0
1	0	0	0	1
0	0	0	1	1

B:

1	0	0	1	0
0	1	0	0	1
0	0	1	0	1

Fig. 6

is simplicial if it and all its neighbors form a clique in G. The algorithm is the following. Select a simplicial vertex in G, say v, and suppose v has w-weight δ. Vertex v is a member of just one maximal clique C in G; assign C a y-component δ, reduce all w-weights of vertices in C by δ, delete v and all other vertices in G having nonpositive weights, and repeat the process with the new graph G' and the new weights w'. (Note that G' is again a rigid-circuit graph, since deleting vertices does not destroy this property. To prove that the algorithm works, construct an independent set in G by making a list of some of the selected simplicial vertices as follows: when simplicial vertex v is selected, add it to the list and delete from the list all vertices that neighbor v in G.)

Rigid circuit graphs and comparability graphs are examples of a class of graphs that are called perfect graphs [1]. For a graph G, let $\gamma(G)$ denote the chromatic number of G (the minimum number of independent sets that cover G), let $\pi(G)$ denote the partition number of G (the minimum number of cliques that cover G), let $\lambda(G)$ denote the clique number of G (the size of a largest clique in G), and let $\omega(G)$ denote the independence number of G (the size of a largest independent set in G). A graph G is γ-<u>perfect</u> if $\gamma(H) = \lambda(H)$ for every (vertex-generated) subgraph H of G; G is π-<u>perfect</u> if $\pi(H) = \omega(H)$ for every (vertex-generated) subgraph H of G; G is <u>perfect</u> if it is both γ-perfect and

π-perfect. Thus a graph is γ-perfect if and only if, for all (0,1)-vectors w, we have (a) the linear program $yB \geq w$, $y \geq 0$, min $1 \cdot y$, has an integer solution y, and (b) $\min 1 \cdot y = \max_{1 \leq i \leq m} a^i \cdot w$, where the clique matrix A has rows $a^1, \ldots, a^m$. Similarly for π-perfection.

In addition to rigid-circuit graphs and comparability graphs, "unimodular" graphs are perfect; that is, if the clique matrix A is totally unimodular, as it is for bipartite graphs, it is known [1] (and follows from Theorem 4.1) that the graph is perfect.

It has been conjectured by Berge that γ-perfection (or π-perfection) implies perfection for a graph. This has been frequently called "the perfect graph conjecture." In this connection we note that the corresponding "plu-perfect graph conjecture" is true. That is, if we define γ-<u>pluperfection</u> to mean that the min-max equality holds strongly for B, A, and <u>pluperfection</u> to mean that the min-max equality holds strongly for both A, B and B, A, then Theorem 4.1 shows that γ-pluperfection implies plu-perfection. Thus to prove the perfect graph conjecture, it suffices to show that γ-perfection implies γ-plu-perfection. For this, it suffices to show that if G is γ-perfect, and if we replace a vertex v in G by two vertices v', v", where v' and v" are joined by an edge and each is joined by an edge to every neighbor of v (i.e., duplicate v and join v to its duplicate), the new graph G' is again γ-perfect.

Example 4. (Independent sets in matroids). Let A be the incidence matrix of the family of independent sets in a matroid on n elements. (For example, A could be the incidence matrix of the family of sub-trees of a graph on n edges.) It has been shown by Edmonds [7] that the inequalities

$$(5.4)\qquad \sum_{i \in S} \xi_i \leq r(S), \quad \text{all } S \subseteq \{1, \ldots, n\},$$

$$(5.5)\qquad \xi_i \geq 0, \quad i \in \{1, \ldots, n\},$$

have as extreme solutions precisely those vectors $x = (\xi_1, \ldots, \xi_n)$ that are incidence vectors of independent sets in the matroid. (In (5.4), $r(S)$ denotes the matroid rank of set S. The inequalities (5.4) are not all essential in general. For instance, S can clearly be restricted to spans (closed sets) in (5.4), but some of these may still yield inessential inequalities.) Thus an anti-blocking matrix for A is the matrix B having a row $b_S = b_S'/r(S)$, where b_S' is the incidence vector of set S, corresponding to each nonempty $S \subseteq \{1, \ldots, n\}$. (We are tacitly assuming that no element has rank zero in the matroid, i.e., our blanket assumption that A has no zero columns.) The min-max equality for A, B does not hold in the strong form, but very nearly so: Edmonds has shown [6] that the best integer answer to the program $yA \geq w$, $y \geq 0$, min $1 \cdot y$, yields

$$\text{(5.6)} \qquad \min 1\cdot y = \langle \max_S w \cdot b_S \rangle ,$$

where $\langle\alpha\rangle$ is the least integer greater than or equal to α.

For an illustration, see Fig. 2. The matrix A' shown there is the incidence matrix of all maximal independent sets (bases) in the cotree matroid (the matroid dual to the tree matroid) of the graph shown there.

Example 5. (Matchings in graphs). Let A be the incidence matrix of the family of all matchings in a graph on n edges. (A matching is a subset of edges, no two on the same vertex.) Here Edmonds has shown [4, 5] that inequalities of two types characterize the convex hull of the rows of A. Let ξ_{ij} be a variable assigned to the edge ij having vertices i and j as ends in the graph G having s vertices. The inequalities can then be written as

$$\text{(5.7)} \qquad \sum_{j\in N_i} \xi_{ij} \leq 1, \quad i = 1, \ldots, s,$$

$$\text{(5.8)} \qquad \sum_{i\in O, j\in O} \xi_{ij} \leq \frac{|O|-1}{2}, \text{ all } O \subseteq \{1, \ldots, s\},$$

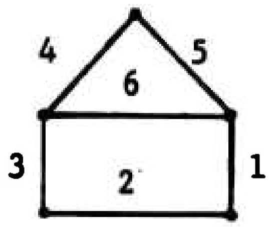
$$\text{(5.9)} \qquad \xi_{ij} \geq 0, \quad \text{all edges } ij.$$

In (5.7), N_i denotes the set of vertices that neighbor i; in (5.8) the subset of vertices O has odd cardinality $|O|$, and the sum is over all edges joining members of O. Thus

(5.7) and (5.8) determine an anti-blocking matrix B of A. (For an illustration, see Fig. 7 below, where we have shown essential rows only.)

Edmonds' proof that (5.7) and (5.8) have just the matchings as extreme solutions is an algorithm for solving the dual programs $Bx \leq 1$, $x \geq 0$, max $w \cdot x$, and $yB \geq w$, $y \geq 0$, min $1 \cdot y$, thereby establishing the min-max equality for B, A.

Note that the best integer answer in the program $yA \geq 1$, $y \geq 0$, min $1 \cdot y$, provides a coloring of the edges of the graph with the least number of colors. Hence the integer form of this problem is unsolved. But if we allow "fractional colorings", i.e., if we consider the linear program $yA \geq w$, $y \geq 0$, min $1 \cdot y$, over the reals or rationals, then Edmonds' result and Theorem 3.1 show what the answer is. For example, one can deduce the following: if G is a tri-valent triply connected graph, then the edges of G can be "fractionally colored" with a "coloring" of total weight three, i.e., min $1 \cdot y = 3$ subject to $yA \geq 1$, $y \geq 0$. (For instance, there are six matchings in the Petersen graph having the property that if each is assigned weight one-half, all edges are covered.)



$$A: \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$B: \left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Fig. 7

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10. ABSTRACT

A geometric duality theory of anti-blocking pairs of polyhedra is developed and applied to a number of problems in extremal combinatorics. It is shown that anti-blocking pairs are characterized by a min-max equality, the analogue of the max-flow min-cut equality for blocking pairs of polyhedra, or by a max-max inequality, the analogue of the length-width inequality for blocking pairs of polyhedra. A main combinatorial result is that if A, B are (0,1)-matrices defining an anti-blocking pair of polyhedra, then the min-max equality holds for both ordered pairs A, B and B, A in a strong integer form. This theorem bears on a well-known unsolved problem in graph theory, the perfect graph conjecture, and in fact proves what might be called the "pluperfect" graph theorem.

11. KEY WORDS

Combinatorics
Network theory
Mathematics
Mathematical programming